

## Quaternions

Quaternions are to vectors as vectors are to points. Where a vector can only translate a point, a quaternion can both scale and rotate a vector. Quaternions were originally created by Sir William Hamilton and can be thought of extending complex numbers into three dimensions,  $\hat{i}, \hat{j}, \hat{k}$ . In computer graphics they are useful to describe rotations and can perform some operations more efficiently than by matrices alone, such as smoothly interpolating between two orientations.

### Definition

A quaternion,  $q$ , is composed of a real part,  $s$ , and a complex vector  $v$ .  
 $q = (s, v)$

### Addition

Addition of two quaternions is performed component wise:  
 $q + q' = (s + s', v + v')$

### Multiplication

Multiplication of quaternions is performed as follows:  
 $qq' = (ss' - v \cdot v', v \times v' + sv' + s'v)$

### Conjugate

The conjugate of a quaternion simply negates the complex component.  
if  $q = (s, v)$  then  $q^* = (s, -v)$

### Magnitude

The magnitude of a quaternion is the product of itself and it's conjugate.  
 $|q| = qq^* = s^2 + v \cdot v = s^2 + x^2 + y^2 + z^2$

### Unit quaternion

A unit quaternion has a magnitude of one. Geometrically this is the set of quaternions that form a unit sphere. They scale the vector by one and perform any possible rotation.

### Inverse of quaternion

$$q^{-1} = \frac{q^*}{|q|}$$

Note: if a quaternion is of unit length, the inverse is simply the conjugate.

## Vector rotation

In order to rotate a vector we must represent both the vector and rotation as quaternions.

A vector can be represented as the complex component of quaternion  $p$  with real part zero.

$$p = (0, v)$$

The rotation of  $\theta$  degrees about axis  $n$  can be represented as follows:

$$q = (\cos(\theta/2), \sin(\theta/2) \cdot n)$$

We rotate by multiplying the quaternion  $p$  by both  $q$  and it's inverse. The order of operations is important as quaternion multiplication is not commutative.

$$r = qpq^{-1}$$

The resulting quaternion  $r$  will have a zero real component and a complex component which is the final rotated vector.

$$r = (0, v)$$

Multiple rotations can be combined by simply multiplying two rotation quaternions together.

$$q' = q_1 q_2$$

## Quaternion to homogeneous rotation matrix

You can directly convert a quaternion into a rotation matrix as follows:

$$M = \begin{vmatrix} 1 - 2(y^2 + z^2) & 2xy - 2sz & 2sy + 2xz & 0 \\ 2xy + 2sz & 1 - 2(x^2 + z^2) & -2sx + 2yz & 0 \\ -2sy + 2xz & 2sx + 2yz & 1 - 2(x^2 + y^2) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Notice the quaternion with  $\theta = 0$  maps to the identity matrix.

$$q = (1, 0) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

## Spherical linear interpolation (SLERP)

Spherical linear interpolation smoothly interpolates between two rotations based on parameter  $u$ .

$$slerp(q_1, q_2, u) = q_1 \frac{\sin((1-u)\omega)}{\sin(\omega)} + q_2 \frac{\sin(u\omega)}{\sin(\omega)} \text{ where } \omega = \arccos(q_1 \cdot q_2) \text{ and } u \in [0, 1]$$

$$slerp(q_1, q_2, 0) = q_1$$

$$slerp(q_1, q_2, 1) = q_2$$