Quaternions

Quaternions are to vectors as vectors are to points. Where a vector can only translate a point, a quaternion can both scale and rotate a vector. Quaternions were originally created by Sir William Hamilton and can be thought of extending complex numbers into three dimensions, $\hat{i}, \hat{j}, \hat{k}$. In computer graphics they are useful to describe rotations and can perform some operations more efficiently than by matrices alone, such as smoothly interpolating between two orientations.

Definition

A quaternion, q, is composed of a real part, s, and a complex vector v.
$q = (s, v)$

Addition

Addition of two quaternions is performed component wise:
$q + q' = (s + s', v + v')$

Multiplication

Multiplication of quaternions is performed as follows:
$qq' = (ss' - v \cdot v', v \times v' + sv' + s'v)$

Conjugate

The conjugate of a quaternion simply negates the complex component.
if $q = (s, v)$ then $q^* = (s, -v)$

Magnitude

The magnitude of a quaternion is the product of itself and it's conjugate.
$|q| = qq^* = s^2 + v \cdot v = s^2 + x^2 + y^2 + z^2$

Unit quaternion

A unit quaternion has a magnitude of one. Geometrically this is the set of quaternions that form a unit sphere. They scale the vector by one and perform any possible rotation.

Inverse of quaternion

$q^{-1} = \frac{q^*}{|q|}$
Note: if a quaternion is of unit length, the inverse is simply the conjugate.

Vector rotation

In order to rotate a vector we must represent both the vector and rotation as quaternions.

A vector can be represented as the complex component of quaternion p with real part zero.
$p = (0, v)$

The rotation of θ degrees about axis n can be represented as follows:
$q = (cos(\theta/2), sin(\theta/2) \cdot n)$

We rotate by multiplying the quaternion p by both q and it's inverse. The order of operations is important as quaternion multiplication is not commutative.
$r = qpq^{-1}$

The resulting quaternion r will have a zero real component and a complex component which is the final rotated vector.
$r = (0, v)$

Multiple rotations can be combined by simply multiplying two rotation quaternions together.
$q' = q_1 q_2$

Quaternion to homogeneous rotation matrix

You can directly convert a quaternion into a rotation matrix as follows:

$$M = \begin{vmatrix} 1-2(y^2+z^2) & 2xy-2sz & 2sy+2xz & 0 \\ 2xy+2sz & 1-2(x^2+z^2) & -2sx+2yz & 0 \\ -2sy+2xz & 2sx+2yz & 1-2(x^2+y^2) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Notice the quaterion with theta = 0 maps to the identity matrix.

$$q = (1, 0) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Spherical linear interpolation (SLERP)

Spherical linear interpolation smoothly interpolates between two rotations based on parameter u.
$slerp(q_1, q_2, u) = q_1 \frac{sin(1-u)\omega}{sin(\omega)} + q_2 \frac{sin(\omega u)}{sin(\omega)}$ where $\omega = \arccos(q_1 \cdot q_2)$ and $u \in [0, 1]$
$slerp(q_1, q_2, 0) = q_1$
$slerp(q_1, q_2, 1) = q_2$